

A FUJITA-TYPE BLOWUP RESULT AND LOW ENERGY SCATTERING FOR A NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this paper we consider the nonlinear Schrödinger equation $iu_t + \Delta u + \kappa|u|^\alpha u = 0$. We prove that if $\alpha < \frac{2}{N}$ and $\Im\kappa < 0$, then every nontrivial H^1 -solution blows up in finite or infinite time. In the case $\alpha > \frac{2}{N}$ and $\kappa \in \mathbb{C}$, we improve the existing low energy scattering results in dimensions $N \geq 7$. More precisely, we prove that if $\frac{8}{N+\sqrt{N^2+16N}} < \alpha \leq \frac{4}{N}$, then small data give rise to global, scattering solutions in H^1 .

1. INTRODUCTION

The main purpose of this article is to prove a Fujita-type blowup result for the nonlinear Schrödinger equation

$$iu_t + \Delta u + \kappa|u|^\alpha u = 0. \quad (1.1)$$

Given an initial value u_0 , the Cauchy problem for (1.1) has the equivalent form

$$u(t) = e^{it\Delta}u_0 + i\kappa \int_0^t e^{i(t-s)\Delta}(|u|^\alpha u)(s) ds. \quad (1.2)$$

As is well known, the Cauchy problem (1.2) is locally well-posed in $H^1(\mathbb{R}^N)$ provided $\alpha < \frac{4}{N-2}$. (See [10].) More precisely, given $u_0 \in H^1(\mathbb{R}^N)$, there exist a maximal existence time $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ and a unique solution $u \in C([0, T_{\max}), H^1(\mathbb{R}^N))$ of (1.2). Moreover, if $T_{\max} < \infty$, then u blows up at T_{\max} in the sense that $\|u(t)\|_{H^1} \rightarrow \infty$ as $t \uparrow T_{\max}$.

Recall that Fujita [7] proved that if $\alpha < \frac{2}{N}$, then all positive solutions of the nonlinear heat equation

$$u_t = \Delta u + |u|^\alpha u \quad (1.3)$$

on \mathbb{R}^N blow up in finite time. In addition, if $\alpha > \frac{2}{N}$, then for initial values sufficiently small in an appropriate sense, the corresponding solution of (1.3) is global in time. See [7]. In the intervening years, this classical result has lead to an extensive literature, see the two survey articles [12, 5]. However, the extensions have always been to parabolic equations.

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It turns out that there is a similar blowup dichotomy for the nonlinear Schrödinger equation (1.1). The blowup part of this dichotomy concerns the case $\Im\kappa < 0$. Indeed, if $\kappa \in \mathbb{R}$, in which case (1.1) becomes the standard nonlinear Schrödinger equation, well-known energy estimates imply that if $\alpha < \frac{4}{N}$, then all H^1 -solutions are global in time and remain bounded. These arguments yield the same result if $\Im\kappa > 0$. On the other hand, we prove that if $\Im\kappa < 0$ and $\alpha < \frac{2}{N}$, there is no global, nontrivial solution of (1.1) that remains bounded in $H^1(\mathbb{R}^N)$. More precisely, we prove the following result.

Theorem 1.1. *Assume $\Im\kappa < 0$ and $\alpha < \frac{2}{N}$. It follows that there exists $\delta > 0$ such that if $u \not\equiv 0$ is a global H^1 solution of (1.1), then*

$$\sup_{0 \leq s \leq t} \|\nabla u(s)\|_{L^2} \geq \delta \|u(0)\|_{L^2}^{\frac{N+2}{N}} t^{\frac{2-N\alpha}{N\alpha}} \quad (1.4)$$

for all $t > 0$. Moreover,

$$t^{-\frac{2-N\alpha}{N\alpha}} \sup_{0 \leq s \leq t} \|\nabla u(s)\|_{L^2} \xrightarrow[t \rightarrow \infty]{} \infty. \quad (1.5)$$

In other words, every nontrivial H^1 -solution blows up in finite or infinite time. We expect that blowup in fact occurs in finite time.

Concerning the global existence part of the dichotomy, it is natural to conjecture that if $\alpha > \frac{2}{N}$, then initial values which are sufficiently small in some norm lead to global solutions which remain bounded and have scattering states in H^1 . This is in fact known in dimension $N = 1, 2, 3$. (See [4, 8, 13].) In higher dimension $N \geq 4$, the best available result seems to be global existence and scattering for small data (i.e. low energy scattering) when $N\alpha + 2\alpha + 2\alpha^2 > 4$, i.e. $\alpha > \alpha_1$ where

$$\alpha_1 = \frac{8}{N + 2 + \sqrt{N^2 + 4N + 36}} \quad (1.6)$$

(See [8, 13].)

The contribution of this paper to the case $\alpha > \frac{2}{N}$ is that we improve the condition $\alpha > \alpha_1$ when $N \geq 7$, to $\alpha > \alpha_2$ with

$$\alpha_2 = \frac{8}{N + \sqrt{N^2 + 16N}} \quad (1.7)$$

Our result in this case is the following.

Theorem 1.2. *Set $X = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx)$ equipped with its natural norm. Let $\kappa \in \mathbb{C}$ and assume*

$$N \geq 3, \quad \alpha_2 < \alpha < \frac{4}{N} \quad (1.8)$$

where α_2 is given by (1.7). Let $u_0 \in X$ satisfy $v_0 \in H^2(\mathbb{R}^N)$ and $|\cdot|v_0 \in H^1(\mathbb{R}^N)$, where $v_0(x) = e^{i\frac{|x|^2}{4}} u_0(x)$. If $\|v_0\|_{H^2}$ is sufficiently small, then the solution of (1.2) is global. Moreover, there exists $u^+ \in X$ such that $e^{-it\Delta} u(t) \rightarrow u^+$ in X as $t \rightarrow \infty$.

Note that $\alpha_2 < \frac{4}{N}$ and behaves as $\frac{4}{N}$ as $N \rightarrow \infty$, not as $\frac{2}{N}$. Thus there is still a significant gap in high dimensions between the conjecture and the known results.

A fundamental technical tool used in the proofs of the above cited results [4, 8, 13] is the Strichartz inequalities. These inequalities involve space-time integrals, where the pair of Lebesgue indices satisfy a certain relationship. Usually, the pairs of Lebesgue indices are *admissible* (see [3, p. 808]), and in particular the low energy

scattering results of [4, 8, 13] use admissible pairs. Strichartz estimates with non-admissible pairs first appeared in [4, Lemma 2.1], but were not used there for low energy scattering. They have subsequently been developed in [11, 6, 16]. In this paper, we use Strichartz estimates with non-admissible pairs along with the low energy scattering argument of [4]. This combination enables us to prove low energy scattering for $\alpha > \alpha_2$.

It is worth noting that the exponent $\alpha = \frac{2}{N}$ is also the critical exponent related to scattering of solutions of (1.1). When $\alpha < \frac{2}{N}$ in dimension $N \geq 2$, it is known that no nonzero solution of (1.1) can be global and scatter in $L^2(\mathbb{R}^N)$. (See Strauss [15], Theorem 3.2 and Example 3.3, p. 68.)

Theorems 1.1 and 1.2 are proved respectively in Sections 2 and 3 below.

2. BLOWUP

The remarkable feature of (1.1) is the identity

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 = -\Im \kappa \int_{\mathbb{R}^N} |u|^{\alpha+2} \quad (2.1)$$

which holds for all $0 \leq t < T_{\max}$. (When $\Im \kappa = 0$, this is the conservation of charge for the standard NLS.) We observe that, were the equation set on a bounded domain with Dirichlet boundary conditions, equation (2.1) together with Hölder's inequality would imply that no H^1 solution can be global (for all $\alpha > 0$), when $\Im \kappa < 0$.

Proof of Theorem 1.1. Let $u(t) \not\equiv 0$ be a global H^1 solution of (1.1). The idea of the proof is to multiply equation (1.1) by a cut-off function, so that the L^2 norm can be controlled by the $L^{\alpha+2}$ norm. We fix the cut-off function $\psi(x) = \nu \theta(|x|)$, where

$$\theta(r) = \begin{cases} 1 & 0 \leq r \leq 1 \\ 2-r & 1 \leq r \leq 2 \\ 0 & r \geq 2 \end{cases} \quad (2.2)$$

and $\nu \in \mathbb{R}$ is such $\|\psi\|_{L^2} = 1$. Given $\lambda > 0$, set

$$\varphi_\lambda(x) = \psi(\lambda x). \quad (2.3)$$

It follows in particular that $\varphi_\lambda \in C_c(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$, $\varphi_\lambda \geq 0$,

$$\|\varphi_\lambda\|_{L^2} = \lambda^{-\frac{N}{2}} \quad \text{and} \quad \|\nabla \varphi_\lambda\|_{L^\infty} = \nu \lambda. \quad (2.4)$$

Multiplying equation (1.1) by $\varphi_\lambda^2 \bar{u}$ and taking the imaginary part, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |u|^2 \varphi_\lambda^2 = 2\Im \int_{\mathbb{R}^N} \bar{u} \varphi_\lambda \nabla u \cdot \nabla \varphi_\lambda - \Im \kappa \int_{\mathbb{R}^N} |u|^{\alpha+2} \varphi_\lambda^2. \quad (2.5)$$

(To be precise, the equation makes sense in H^{-1} , so we take the $H^{-1} - H^1$ duality bracket of the equation with $\varphi_\lambda^2 \bar{u} \in H^1$.) Set

$$f_\lambda(t) = \|u \varphi_\lambda\|_{L^2} \quad (2.6)$$

and

$$K_t = \|\nabla u\|_{L^\infty((0,t),L^2)}. \quad (2.7)$$

It follows from Hölder's inequality and (2.4) that

$$f_\lambda(t)^{\alpha+2} \leq \|\varphi_\lambda\|_{L^2}^\alpha \int_{\mathbb{R}^N} |u|^{\alpha+2} \varphi_\lambda^2 = \lambda^{-\frac{N\alpha}{2}} \int_{\mathbb{R}^N} |u|^{\alpha+2} \varphi_\lambda^2,$$

so that

$$\int_{\mathbb{R}^N} |u|^{\alpha+2} \varphi_\lambda^2 \geq \lambda^{\frac{N\alpha}{2}} f_\lambda(t)^{\alpha+2}. \quad (2.8)$$

Moreover, we deduce from Hölder's inequality, (2.4), (2.6) and (2.7) that

$$\left| \Im \int_{\mathbb{R}^N} \bar{u} \varphi_\lambda \nabla u \cdot \nabla \varphi_\lambda \right| \leq \nu \lambda K_t f_\lambda(t). \quad (2.9)$$

Consequently, (2.5), (2.8) and (2.9) yield

$$f'_\lambda \geq -2\nu \lambda K_t - \Im \kappa \lambda^{\frac{N\alpha}{2}} f_\lambda^{\alpha+1} \geq -2\nu \lambda K_T - \Im \kappa \lambda^{\frac{N\alpha}{2}} f_\lambda^{\alpha+1}, \quad (2.10)$$

for all $0 < t \leq T < \infty$, where we used the property that K_t is nondecreasing in t in the last inequality. Therefore, if

$$f_\lambda(0)^{\alpha+1} \geq 4(-\Im \kappa)^{-1} \lambda^{\frac{2-N\alpha}{2}} \nu K_T, \quad (2.11)$$

it follows that f_λ is increasing on $(0, T)$, and

$$f'_\lambda \geq \frac{-\Im \kappa}{2} \lambda^{\frac{N\alpha}{2}} f_\lambda^{\alpha+1} \quad (2.12)$$

on $(0, T)$. Equation (2.12) implies that f_λ must blow up before the finite time $\frac{2}{-\Im \kappa \alpha} \lambda^{-\frac{N\alpha}{2}} f_\lambda(0)^{-\alpha}$. Therefore,

$$T \leq \frac{2}{-\Im \kappa \alpha} \lambda^{-\frac{N\alpha}{2}} f_\lambda(0)^{-\alpha}. \quad (2.13)$$

Note that $f_\lambda(0)$ is a nonincreasing function of $\lambda > 0$ and

$$f_\lambda(0) = \left(\int_{\mathbb{R}^N} |u(0, x)|^2 \psi^2(\lambda x) dx \right)^{\frac{1}{2}} \xrightarrow{\begin{cases} 0 & \text{as } \lambda \uparrow \infty \\ \|u(0, \cdot)\|_{L^2} & \text{as } \lambda \downarrow 0. \end{cases}} \quad (2.14)$$

We first show that

$$K_T \xrightarrow[T \rightarrow \infty]{} \infty. \quad (2.15)$$

Indeed, suppose by contradiction that K_T is bounded as $T \rightarrow \infty$. It follows from (2.14) that we can choose $\tilde{\lambda} > 0$ sufficiently small so that (2.11) with $\lambda = \tilde{\lambda}$ holds for all $T > 0$. We deduce from (2.13) that $T \leq \frac{2}{-\Im \kappa \alpha} \tilde{\lambda}^{-\frac{N\alpha}{2}} f_{\tilde{\lambda}}(0)^{-\alpha}$ for all $T > 0$. This is absurd, proving (2.15).

We next prove (1.4). Fix $\lambda_0 > 0$ such that

$$f_{\lambda_0}(0) = \frac{1}{2} \|u(0, \cdot)\|_{L^2}. \quad (2.16)$$

Note that this is possible by (2.14). It follows from (2.15) that if $T > 0$ is sufficiently large, $\lambda = \lambda(T)$ defined by

$$f_{\lambda_0}(0)^{\alpha+1} = 4(-\Im \kappa)^{-1} \lambda(T)^{\frac{2-N\alpha}{2}} K_T, \quad (2.17)$$

satisfies

$$\lambda(T) \leq \lambda_0. \quad (2.18)$$

Since $f_\lambda(0)$ is a nonincreasing function of λ , we deduce from (2.17)-(2.18) that (2.11) holds with $\lambda = \lambda(T)$. Therefore, it follows from (2.13) that

$$T \leq \frac{2}{-\Im \kappa \alpha} \lambda(T)^{-\frac{N\alpha}{2}} f_{\lambda(T)}(0)^{-\alpha}. \quad (2.19)$$

Using again (2.18), we deduce from (2.19) that

$$T \leq \frac{2}{-\Im\kappa\alpha} \lambda(T)^{-\frac{N\alpha}{2}} f_{\lambda_0}(0)^{-\alpha}. \quad (2.20)$$

Since (2.17) implies

$$\lambda(T)^{-\frac{2-N\alpha}{2}} = \frac{4}{-\Im\kappa} K_T f_{\lambda_0}(0)^{-(\alpha+1)} \quad (2.21)$$

formulas (2.20) and (2.21) yield

$$\begin{aligned} T^{\frac{2-N\alpha}{N\alpha}} &\leq \frac{4}{-\Im\kappa} \left(\frac{2}{-\Im\kappa\alpha} f_{\lambda_0}(0)^{-\alpha} \right)^{\frac{2-N\alpha}{N\alpha}} f_{\lambda_0}(0)^{-(\alpha+1)} K_T \\ &= 2^{\frac{2+N\alpha}{N\alpha}} (-\Im\kappa)^{-\frac{2}{N\alpha}} \alpha^{-\frac{2-N\alpha}{N\alpha}} f_{\lambda_0}(0)^{-\frac{N+2}{N}} K_T. \end{aligned} \quad (2.22)$$

Inequality (1.4) now follows from (2.22) and (2.16).

We finally prove (1.5). Given $T > 0$, it follows from (2.14) that there exists a unique $\mu(T) > 0$ such that

$$f_{\mu(T)}(0)^{\alpha+1} = 4(-\Im\kappa)^{-1} \mu(T)^{\frac{2-N\alpha}{2}} K_T. \quad (2.23)$$

Since K_T is a nondecreasing function of T , it follows from (2.23) that the map $T \mapsto f_{\mu(T)}(0)^{\alpha+1} \mu(T)^{-\frac{2-N\alpha}{2}}$ is also nondecreasing. On the other hand, $f_\mu(0)$ is a nonincreasing function of μ , so that the map $\mu \mapsto f_\mu(0)^{\alpha+1} \mu^{-\frac{2-N\alpha}{2}}$ is decreasing, so we conclude that the map $T \mapsto \mu(T)$ is nonincreasing. Since $f_\mu(0) \leq \|u(0)\|_{L^2}$ for all $\mu > 0$ by (2.14) and $K_T \geq \delta \|u(0)\|_{L^2}^{\frac{N+2}{N}} T^{\frac{2-N\alpha}{N\alpha}}$ by (1.4) we deduce from (2.23) that

$$\mu(T) \leq (4(-\Im\kappa)^{-1} \delta)^{-\frac{2}{2-N\alpha}} \|u(0)\|_{L^2}^{-\frac{2}{N}} T^{-\frac{2}{N\alpha}} \xrightarrow[T \rightarrow \infty]{} 0. \quad (2.24)$$

We deduce in particular from (2.14) and (2.24) that $f_{\mu(T)}(0) \rightarrow \|u(0)\|_{L^2}$ as $T \rightarrow \infty$ so that by (2.23)

$$\mu(T)^{\frac{2-N\alpha}{2}} K_T \xrightarrow[T \rightarrow \infty]{} \frac{\|u(0)\|_{L^2}^{\alpha+1} (-\Im\kappa)}{4}. \quad (2.25)$$

Moreover, it follows from (2.23) that (2.11) is satisfied with $\lambda = \mu(T)$, so that (2.12) holds, i.e.

$$f'_{\mu(T)} \geq \frac{-\Im\kappa}{2} \mu(T)^{\frac{N\alpha}{2}} f_{\mu(T)}^{\alpha+1} \quad (2.26)$$

for all $0 < t < T$. Integrating the above differential inequality on $(T/2, T)$ and using (2.25), we obtain

$$f_{\mu(T)}(T/2)^{-\alpha} - f_{\mu(T)}(T)^{-\alpha} \geq \frac{-\Im\kappa\alpha}{4} \mu(T)^{\frac{N\alpha}{2}} T \geq \eta K_T^{-\frac{N\alpha}{2-N\alpha}} T \quad (2.27)$$

for $T \geq 2$, with $\eta > 0$. Next, recall that $\mu(t)$ is a nonincreasing function of t , and that the map $\lambda \mapsto f_\lambda(t)$ is a nonincreasing function of λ , so that

$$f_{\mu(s)}(\tau) \leq f_{\mu(t)}(\tau) \quad (2.28)$$

for all $\tau > 0$ and $0 < s < t$. Therefore, letting $s = \tau = T/2$ and $t = T$, we see that $f_{\mu(T/2)}(T/2)^{-\alpha} \geq f_{\mu(T)}(T/2)^{-\alpha}$ and it follows from (2.27) that

$$f_{\mu(T/2)}(T/2)^{-\alpha} - f_{\mu(T)}(T)^{-\alpha} \geq \eta K_T^{-\frac{N\alpha}{2-N\alpha}} T \quad (2.29)$$

for $T \geq 2$. Next, we deduce from (2.28) and the fact that the map $\tau \mapsto f_{\mu(t)}(\tau)$ is increasing on $(0, t)$, that

$$f_{\mu(s)}(s) \leq f_{\mu(t)}(s) \leq f_{\mu(t)}(t) \quad (2.30)$$

for all $0 < s < t$. Thus we see that the map $t \mapsto f_{\mu(t)}(t)$ is nondecreasing; and so the map $t \mapsto f_{\mu(t)}(t)^{-\alpha}$ is nonincreasing, so it has a limit as $t \rightarrow \infty$. Letting $T \rightarrow \infty$ in (2.29), we deduce that $K_T^{-\frac{N\alpha}{2-N\alpha}} T \rightarrow 0$ as $T \rightarrow \infty$, which is the desired result. \square

Remark 2.1. Under the assumption

$$-\Im\kappa \geq \frac{\alpha}{2\sqrt{\alpha+1}} |\Re\kappa| \quad (2.31)$$

we may replace $\sup_{0 \leq s \leq t} \|\nabla u(s)\|_{L^2}$ by $\|\nabla u(t)\|_{L^2}$ in estimates (1.4) and (1.5) of Theorem 1.1. Indeed, it follows from (2.31) that $\|\nabla u(t)\|_{L^2}$ is a nondecreasing function of t . To see this, note that for a solution of (1.1) we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 = \Re \left(i\kappa \int_{\mathbb{R}^N} \nabla(|u|^\alpha u) \cdot \nabla \bar{u} \right) \stackrel{\text{def}}{=} A. \quad (2.32)$$

Since

$$\nabla(|u|^\alpha u) = \frac{\alpha+2}{2} |u|^\alpha \nabla u + \frac{\alpha}{2} |u|^{\alpha-2} u^2 \nabla \bar{u}, \quad (2.33)$$

we see that

$$\nabla(|u|^\alpha u) \cdot \nabla \bar{u} = \frac{\alpha+2}{2} |u|^\alpha |\nabla u|^2 + \frac{\alpha}{2} |u|^{\alpha-2} u^2 (\nabla \bar{u})^2. \quad (2.34)$$

It follows that

$$\begin{aligned} \Re[i\kappa \nabla(|u|^\alpha u) \cdot \nabla \bar{u}] &= -\Im\kappa \frac{\alpha+2}{2} |u|^\alpha |\nabla u|^2 + \frac{\alpha}{2} \Re[i\kappa |u|^{\alpha-2} u^2 (\nabla \bar{u})^2] \\ &\geq \left(-\frac{\alpha+2}{2} \Im\kappa - \frac{\alpha}{2} |\kappa| \right) |u|^\alpha |\nabla u|^2. \end{aligned} \quad (2.35)$$

This shows that $A \geq 0$ provided (2.31) holds. The above calculations are justified if u is an H^2 solution. The result follows by approximation, regularity, and continuous dependence. (All these properties are established in [10].) Note that (2.31) is identical to condition (2.2) in [14].

Remark 2.2. Under the assumptions of Theorem 1.1, we do not know whether or not there exists a global H^1 solution of (1.1). In fact, if such a solution does exist, it would necessarily have a stronger dispersion than the solutions of the linear Schrödinger equation. Indeed, suppose $u \not\equiv 0$ is a global H^1 solution of (1.1) and let $R(t)$ satisfy

$$\int_{\{|x| \leq R(t)\}} |u(t, x)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^N} |u(t, x)|^2 dx.$$

We claim that

$$\limsup_{t \rightarrow \infty} t^{-\frac{2}{N\alpha}} R(t) = \infty. \quad (2.36)$$

To see this, observe that by Hölder's inequality and the definition of $R(t)$,

$$\int_{\mathbb{R}^N} |u|^{\alpha+2} \geq 2^{-\frac{\alpha+2}{2}} \omega_N^{-\frac{\alpha}{2}} R^{-\frac{N\alpha}{2}} f(t)^{\frac{\alpha+2}{2}} \quad (2.37)$$

where $f(t) = \int_{\mathbb{R}^N} |u|^2$ and ω_N is the measure of the unit ball of \mathbb{R}^N . It follows from (1.1) and (2.37) that

$$f' \geq -\Im \kappa 2^{-\frac{\alpha}{2}} \omega_N^{-\frac{\alpha}{2}} R^{-\frac{N\alpha}{2}} f^{\frac{\alpha+2}{2}}.$$

Therefore, $\int_0^\infty R(t)^{-\frac{N\alpha}{2}} dt < \infty$, which yields (2.36). On the other hand, let $\tilde{u}(t) = e^{it\Delta} u_0$ where $u_0 \in H^1(\mathbb{R}^N)$, $u_0 \neq 0$. Multiplying the equation $i\tilde{u}_t + \Delta \tilde{u} = 0$ by $\psi_M \bar{u}$, where $\psi_M(x) = \min\{\frac{x}{M}, 1\}$, we obtain

$$\int_{\mathbb{R}^N} \psi_M |\tilde{u}|^2 \leq \int_{\mathbb{R}^N} \psi_M |u_0|^2 + \frac{2t}{M} \|u_0\|_{L^2} \|\nabla u_0\|_{L^2}. \quad (2.38)$$

(Cf. [9, Lemma 5.4].) Given $t > 0$, we substitute $M = at$ in (2.38) with $a = 16 \frac{\|\nabla u_0\|_{L^2}}{\|u_0\|_{L^2}}$. Since $\psi_M \geq 1_{\{|x| > M\}}$, this yields

$$\int_{\{|x| > at\}} |\tilde{u}|^2 \leq \int_{\mathbb{R}^N} \psi_{at} |u_0|^2 + \frac{1}{8} \|u_0\|_{L^2}^2.$$

Furthermore, $\int_{\mathbb{R}^N} \psi_{at} |u_0|^2 \rightarrow 0$ as $t \rightarrow \infty$ by dominated convergence, and so

$$\int_{\{|x| > at\}} |\tilde{u}|^2 \leq \frac{1}{4} \|u_0\|_{L^2}^2 \quad (2.39)$$

for t large. Therefore,

$$\int_{\{|\tilde{x}| < at\}} |u|^2 \geq \frac{3}{4} \|u_0\|_{L^2}^2 \quad (2.39)$$

for t large. In particular, if $\tilde{R}(t)$ satisfies

$$\int_{\{|x| \leq \tilde{R}(t)\}} |\tilde{u}(t, x)|^2 dx = \frac{1}{2} \int_{\mathbb{R}^N} |\tilde{u}(t, x)|^2 dx = \frac{1}{2} \|u_0\|_{L^2}^2,$$

then $\tilde{R}(t) \leq at$ for t large. Comparing with (2.36), we see that u has a stronger dispersion than \tilde{u} as $t \rightarrow \infty$.

Remark 2.3. If we look for solutions of (1.1) of the form

$$u(t, x) = \rho(t) e^{i \frac{|x|^2}{4(t+t_0)}},$$

where $t_0 > 0$ is given, then ρ must satisfy

$$\rho' = -\frac{N}{2(t+t_0)} \rho - i\kappa |\rho|^\alpha \rho.$$

Setting $z = (t+t_0)^{\frac{N}{2}} \rho$, we get to the equation

$$z' = -i\kappa (t+t_0)^{-\frac{N\alpha}{2}} |z|^\alpha z.$$

Multiplying the equation by \bar{z} and taking the real part, one easily gets to

$$\frac{1}{\alpha |z(t)|^\alpha} = \frac{1}{\alpha |z(0)|^\alpha} + \Im \kappa \int_0^t \frac{ds}{(s+t_0)^{\frac{N\alpha}{2}}}. \quad (2.40)$$

If $\alpha > \frac{2}{N}$, then the integral on the right-hand side of (2.40) is convergent, and we see that if $|z(0)|$ is sufficiently small so that

$$\frac{1}{\alpha |z(0)|^\alpha} \geq -\Im \kappa \int_0^\infty \frac{ds}{(s+t_0)^{\frac{N\alpha}{2}}},$$

then the solution is global; and if $|z(0)|$ is larger, then the solution blows up in finite time. On the other hand, if $\alpha \leq \frac{2}{N}$, then the integral on the right-hand side of (2.40) is divergent. Therefore, for every $z(0)$, the solution blows up at the finite time T given by

$$\frac{1}{\alpha|z(0)|^\alpha} = -\Im\kappa \int_0^T \frac{ds}{(s+t_0)^{\frac{N\alpha}{2}}}.$$

Thus we see that the exponent $\alpha = \frac{2}{N}$ is critical.

3. LOW ENERGY SCATTERING

To prove Theorem 1.2, we first prove a local existence result for small data for the following equation

$$v(t) = e^{it\Delta} v_0 + \int_0^t h(s) e^{i(t-s)\Delta} (|v|^\alpha v)(s) ds \quad (3.1)$$

where

$$h(t) = i\kappa(1-t)^{-\frac{4-N\alpha}{2}} \quad (3.2)$$

for $0 \leq t < 1$. As we will see, equation (3.1) is equivalent to equation (1.1) via the pseudo-conformal transformation. Before stating the result, we introduce some notation. We assume (1.8) and we set

$$\rho = \frac{N(\alpha+2)}{N+\alpha}, \quad \gamma = \frac{4(\alpha+2)}{\alpha(N-2)} = \frac{4\rho}{N(\rho-2)}. \quad (3.3)$$

It is not difficult to show that

$$2 < \rho < \frac{2N}{N-2}, \quad N > \rho \quad (3.4)$$

and that (γ, ρ) is an admissible pair, i.e. $\frac{2}{\gamma} = N(\frac{1}{2} - \frac{1}{\rho})$ (see [3, Proposition 1.5 (ii)]). Note also that $\alpha > \alpha_2$, which implies $N\alpha^2 + N\alpha > 4$, so that

$$\frac{4 - (N-4)\alpha}{2(\alpha+2)} > \frac{4 - N\alpha}{2} \quad (3.5)$$

Theorem 3.1. *Suppose $N \geq 3$, $\kappa \in \mathbb{C}$ and $\alpha_2 < \alpha < \frac{4}{N}$ where α_2 is given by (1.7). Fix*

$$a \geq \gamma \quad (3.6)$$

sufficiently large so that

$$\frac{4 - (N-4)\alpha}{2(\alpha+2)} - \frac{\alpha}{a} > \frac{4 - N\alpha}{2}. \quad (3.7)$$

(The existence of a is guaranteed by (3.5).) There exists $\delta > 0$ such that if

$$v_0 \in X \quad (3.8)$$

$$(1 + |\cdot|)e^{it\Delta} v_0 \in L^a((0, 1), L^\rho(\mathbb{R}^N)) \quad (3.9)$$

$$e^{it\Delta} \nabla v_0 \in L^a((0, 1), L^\rho(\mathbb{R}^N)) \quad (3.10)$$

$$\|\nabla e^{it\Delta} v_0\|_{L^a((0,1), L^\rho)} \leq \delta \quad (3.11)$$

then there exists a solution $v \in C([0, 1], X)$ of (3.1).

Proof of Theorem 3.1. We define \tilde{a} by

$$\frac{1}{a} + \frac{1}{\tilde{a}} = \frac{2}{\gamma} \quad (3.12)$$

and we recall the following Strichartz-type estimate for non-admissible pairs

$$\left\| \int_0^{\cdot} e^{i(\cdot-s)\Delta} f(s) ds \right\|_{L^a((0,1), L^\rho)} \leq C \|f\|_{L^{\tilde{a}'}((0,1), L^{\rho'})}. \quad (3.13)$$

(See [4, Lemma 2.1].) If μ is defined by

$$\frac{1}{\mu} = \frac{4 - (N - 4)\alpha}{2(\alpha + 2)} - \frac{\alpha}{a} < \frac{4 - (N - 4)\alpha}{2(\alpha + 2)} < 1 \quad (3.14)$$

then it follows from (3.7) and (3.2) that

$$h \in L^\mu(0,1). \quad (3.15)$$

Next, we deduce from Sobolev's inequality that

$$\|v\|_{L^{\frac{\alpha\rho}{\rho-2}}} = \|v\|_{L^{\frac{N\rho}{N-\rho}}} \leq C \|\nabla v\|_{L^\rho} \quad (3.16)$$

and so by Hölder's inequality,

$$\| |v|^\alpha w \|_{L^{\rho'}} \leq \|v\|_{L^{\frac{\alpha\rho}{\rho-2}}}^\alpha \|w\|_{L^\rho} \leq C \|\nabla v\|_{L^\rho}^\alpha \|w\|_{L^\rho}. \quad (3.17)$$

It easily follows from (3.17) that

$$\| |v|^\alpha v \|_{L^{\rho'}} \leq C \|\nabla v\|_{L^\rho}^\alpha \|v\|_{L^\rho} \quad (3.18)$$

$$\| | \cdot | |v|^\alpha v \|_{L^{\rho'}} \leq C \|\nabla v\|_{L^\rho}^\alpha \| | \cdot | v \|_{L^\rho} \quad (3.19)$$

$$\| \nabla (|v|^\alpha v) \|_{L^{\rho'}} \leq C \|\nabla v\|_{L^\rho}^{\alpha+1} \quad (3.20)$$

$$\| |u|^\alpha u - |v|^\alpha v \|_{L^{\rho'}} \leq C (\| \nabla u \|_{L^\rho}^\alpha + \| \nabla v \|_{L^\rho}^\alpha) \|u - v\|_{L^\rho}. \quad (3.21)$$

We observe, by (3.14) and (3.12), that

$$\frac{1}{\tilde{a}'} = \frac{1}{\mu} + \frac{\alpha + 1}{a}. \quad (3.22)$$

Again using Hölder's inequality, but with the time integrals, we deduce from (3.22), along with respectively (3.18), (3.19), (3.20) and (3.21), that

$$\|h|v|^\alpha v\|_{L^{\tilde{a}'}((0,1), L^{\rho'})} \leq C \|h\|_{L^\mu(0,1)} \|\nabla v\|_{L^a((0,1), L^\rho)}^\alpha \|v\|_{L^a((0,1), L^\rho)} \quad (3.23)$$

$$\|h| \cdot | |v|^\alpha v\|_{L^{\tilde{a}'}((0,1), L^{\rho'})} \leq C \|h\|_{L^\mu(0,1)} \|\nabla v\|_{L^a((0,1), L^\rho)}^\alpha \| | \cdot | v \|_{L^a((0,1), L^\rho)} \quad (3.24)$$

$$\| \nabla (h|v|^\alpha v) \|_{L^{\tilde{a}'}((0,1), L^{\rho'})} \leq C \|h\|_{L^\mu(0,1)} \|\nabla v\|_{L^a((0,1), L^\rho)}^{\alpha+1} \quad (3.25)$$

and

$$\begin{aligned} \|h(|u|^\alpha u - |v|^\alpha v)\|_{L^{\tilde{a}'}((0,1), L^{\rho'})} &\leq C \|h\|_{L^\mu(0,1)} \\ &\times (\| \nabla u \|_{L^a((0,1), L^\rho)}^\alpha + \| \nabla v \|_{L^a((0,1), L^\rho)}^\alpha) \|u - v\|_{L^a((0,1), L^\rho)}. \end{aligned} \quad (3.26)$$

We construct the solution v of (1.2) by a contraction mapping argument in the set $\mathcal{E}_{\delta, M}$ defined for $\delta, M > 0$ by

$$\begin{aligned} \mathcal{E}_{\delta, M} = \{v \in L^a((0,1), W^{1,\rho}(\mathbb{R}^N)); & | \cdot | v \in L^a((0,1), L^\rho(\mathbb{R}^N)), \\ & \|v\|_{L^a((0,1), L^\rho)} \leq M, \| | \cdot | v \|_{L^a((0,1), L^\rho)} \leq M \text{ and } \|\nabla v\|_{L^a((0,1), L^\rho)} \leq \delta\}. \end{aligned} \quad (3.27)$$

We set $d(v, w) = \|v - w\|_{L^a(0,1), L^\rho}$ so that $(\mathcal{E}_{\delta, M}, d)$ is a complete metric space. Fix $v_0 \in X$ and, given $v \in \mathcal{E}_{\delta, M}$, let $\mathcal{I}(v)$ and $\Phi(v)$ be defined by

$$\mathcal{I}(v)(t) = \int_0^t h(s) e^{i(t-s)\Delta} (|v|^\alpha v)(s) ds \quad (3.28)$$

$$\Phi(v)(t) = e^{it\Delta} v_0 + \mathcal{I}(v)(t). \quad (3.29)$$

It follows from (3.23), (3.25), (3.26) and the estimate (3.13) that, for some constant C independent of δ , M and $v, w \in \mathcal{E}_{\delta, M}$,

$$\|\mathcal{I}(v)\|_{L^a((0,1), L^\rho)} \leq C \|h\|_{L^\mu(0,1)} \delta^\alpha M \quad (3.30)$$

$$\|\nabla \mathcal{I}(v)\|_{L^a((0,1), L^\rho)} \leq C \|h\|_{L^\mu(0,1)} \delta^{\alpha+1} \quad (3.31)$$

$$\|\mathcal{I}(v) - \mathcal{I}(w)\|_{L^a((0,1), L^\rho)} \leq C \|h\|_{L^\mu(0,1)} \delta^\alpha d(v, w). \quad (3.32)$$

Next, we estimate the weighted norm. We observe that

$$xe^{i\tau\Delta} = e^{i\tau\Delta}(x - 2i\tau\nabla) \quad (3.33)$$

for all $\tau \in \mathbb{R}$. Therefore

$$\begin{aligned} x\mathcal{I}(v)(t) &= \int_0^t h(s) e^{i(t-s)\Delta} (x - 2i(t-s)\nabla) |v|^\alpha v \\ &= \int_0^t h(s) e^{i(t-s)\Delta} x |v|^\alpha v - 2i \int_0^t h(s) (t-s) e^{i(t-s)\Delta} \nabla (|v|^\alpha v) \end{aligned} \quad (3.34)$$

and we deduce from (3.24), (3.25) and (3.13) that

$$\| |\cdot| \mathcal{I}(v) \|_{L^a((0,1), L^\rho)} \leq C \|h\|_{L^\mu(0,1)} \delta^\alpha M. \quad (3.35)$$

We now set

$$M = 2 \max\{\|e^{it\Delta} v_0\|_{L^a((0,1), L^\rho)}, \| |\cdot| e^{it\Delta} v_0 \|_{L^a((0,1), L^\rho)}\} \quad (3.36)$$

$$\delta = 2 \|\nabla v_0\|_{L^a((0,1), L^\rho)}. \quad (3.37)$$

It follows from (3.30), (3.35) and (3.31) that if δ is sufficiently small, then

$$\|\mathcal{I}(v)\|_{L^a((0,1), L^\rho)} \leq \frac{M}{2}, \quad \| |\cdot| \mathcal{I}(v) \|_{L^a((0,1), L^\rho)} \leq \frac{M}{2}, \quad \|\nabla \mathcal{I}(v)\|_{L^a((0,1), L^\rho)} \leq \frac{\delta}{2}.$$

Applying (3.36)-(3.37) and (3.28)-(3.29), we deduce that $\Phi : \mathcal{E}_{\delta, M} \rightarrow \mathcal{E}_{\delta, M}$. Moreover, assuming δ possibly smaller, it follows from (3.32) that Φ is a strict contraction on $\mathcal{E}_{\delta, M}$. By Banach's fixed point theorem, Φ has a fixed point $v \in \mathcal{E}_{\delta, M}$, which is a solution of (1.1).

To complete the proof, it remains to show that $v \in C([0, 1], X)$. For this, we observe that by (3.6) and (3.12) we have $\tilde{\alpha} \leq \gamma$, so that $\tilde{\alpha}' \geq \gamma'$. Therefore, estimates (3.23), (3.24) and (3.25), and the fact that $v \in \mathcal{E}_{\delta, M}$ imply that

$$\begin{aligned} \|h|v|^\alpha v\|_{L^{\gamma'}((0,1), L^{\rho'})} &\leq C \delta^\alpha M \\ \|h| |\cdot| v|^\alpha v\|_{L^{\gamma'}((0,1), L^{\rho'})} &\leq C \delta^\alpha M \\ \|\nabla(h|v|^\alpha v)\|_{L^{\gamma'}((0,1), L^{\rho'})} &\leq C \delta^{\alpha+1} \end{aligned}$$

It now follows from the standard Strichartz estimates (i.e., with admissible pairs, see e.g. [2, Theorem 2.2.3 (ii)]) that $\mathcal{I}(v) \in C([0, 1], X)$. Since $v = \mathcal{I}(v) + e^{it\Delta} v_0$ and $v_0 \in X$, this completes the proof. \square

Remark 3.2. The conditions (3.8)–(3.11) are satisfied under some stronger, but more familiar, conditions. Indeed, set

$$s = N \left(\frac{1}{2} - \frac{1}{\rho} \right) - \frac{2}{a} \quad (3.38)$$

so that $0 \leq s < 1$ by (3.6) and (3.4). Setting $\frac{1}{\rho} = \frac{1}{\rho} + \frac{s}{N}$, we see that $(a, \tilde{\rho})$ is an admissible pair, so that by Strichartz's estimates (see e.g. [3, Theorem 2.2 (i)]) $\|e^{it\Delta} \varphi\|_{L^a(\mathbb{R}, \dot{H}^{s, \tilde{\rho}})} \leq C \|\varphi\|_{\dot{H}^s}$, where $\dot{H}^{s, p}$ and $\dot{H}^s = \dot{H}^{s, 2}$ are the homogeneous Sobolev spaces (see e.g. [1, Section 6.3]). Using Sobolev's embedding, we deduce that

$$\|e^{it\Delta} \varphi\|_{L^a(\mathbb{R}, L^\rho)} \leq C \|\varphi\|_{\dot{H}^s}.$$

Therefore, conditions (3.8)–(3.11) are satisfied provided $v_0 \in X$, $v_0 \in \dot{H}^s(\mathbb{R}^N)$, $|\cdot|v_0 \in \dot{H}^s(\mathbb{R}^N)$, $\nabla v_0 \in \dot{H}^s(\mathbb{R}^N)$, and the smallness condition is on the norm $\|\nabla v_0\|_{\dot{H}^s}$.

Proof of Theorem 1.2. Let u_0 be as in Theorem 1.2. In particular, v_0 defined by $v_0(x) = e^{i\frac{|x|^2}{4}} u_0(x)$ satisfies $v_0 \in X$. Moreover, if s is defined by (3.38), then $s \in [0, 1)$, so that $v_0 \in \dot{H}^s(\mathbb{R}^N)$, $|\cdot|v_0 \in \dot{H}^s(\mathbb{R}^N)$, $\nabla v_0 \in \dot{H}^s(\mathbb{R}^N)$. Therefore, it follows from Remark 3.2 that $e^{it\Delta} v_0 \in L^a((0, 1), L^\rho(\mathbb{R}^N))$, $|\cdot|e^{it\Delta} v_0 \in L^a((0, 1), L^\rho(\mathbb{R}^N))$, $e^{it\Delta} \nabla v_0 \in L^a((0, 1), L^\rho(\mathbb{R}^N))$ and $\|e^{it\Delta} \nabla v_0\|_{L^a((0, 1), L^\rho)} \leq \|v_0\|_{H^2}$. Thus we see that if $\|v_0\|_{H^2}$ is sufficiently small, then v_0 satisfies the assumptions of Theorem 3.1. Let $v \in C([0, 1], X)$ be the corresponding solution of (3.1). Following [4], we apply the pseudo-conformal transformation. More precisely, let the variables $(s, y) \in [0, \infty) \times \mathbb{R}^N$ be defined by

$$s = \frac{t}{1-t}, \quad y = \frac{x}{1-t}$$

for $(t, x) \in [0, 1) \times \mathbb{R}^N$. We define u on $[0, \infty) \times \mathbb{R}^N$ by

$$u(s, y) = (1-t)^{N/2} e^{i\frac{|x|^2}{4(1-t)}} v(t, x).$$

It follows that $u \in C([0, \infty), X)$, and is a solution of (1.2) on $[0, \infty)$. Finally, since $v \in C([0, 1], X)$, it follows from Proposition 3.14 in [4] that there exists $u^+ \in X$ such that $e^{-is\Delta} u(s) \rightarrow u^+$ in X as $s \rightarrow \infty$. This completes the proof. \square

Remark 3.3. The conditions on the initial value u_0 in Theorem 1.2 are stronger than the conditions that are actually used in the proof. These latter conditions are expressed in terms of $v_0 = e^{i\frac{|x|^2}{4}} u_0$ in formulas (3.8)–(3.11). Intermediate conditions are given in Remark 3.2.

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